HEAT TRANSFER IN CHANNELS IN THE LAMINAR FLOW OF A NON-NEWTONIAN LIQUID WITH SLIP

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We consider the case of quasi-isothermal heat transfer in the laminar flow of a non-Newtonian medium with slip in circular and plane channels. The calculation is based on the simultaneous solution of the equations of energy, motion, and a rheological relation with the specific boundary conditions which takes into account discontinuity in the velocity and temperature on the heat-transfer surface.

Experiments show that under normal conditions boundary-layer slip can be observed in media which are structurally viscous [1,2], viscoelastic [3,4], viscoplastic [5,6], etc. Motion with slip can also be observed in ordinary Newtonian liquids when they move in very fine capillaries [7]. There is information that in the flow of blood in capillaries and arterioles the flow rate per second in a capillary can exceed that calculated by Poiseuille's equation (ignoring slip) by 2-4 times.

Discontinuity in velocity and temperature at the heat-transfer surface can arise as a result of crystallization, cross-linking of polymer chains, or the presence of elastic waves which absorb some of the momentum and heat and also as a result of the detachment of the liquid from the surface.

The equation for the energy in a steady continuous axisymmetric laminar flow at Péclet numbers $P > 10$ has the form

$$w \frac{\partial t}{\partial x} = \frac{a}{r} \frac{\partial}{\partial r}\left(r \frac{\partial t}{\partial r}\right) \tag{1}$$

Here w is the flow velocity, t is the flow temperature, a is the coefficient of thermal diffusivity.

Since non-Newtonian media displaying slip are characterized by Prandtl numbers $\sigma = \nu a^{-1} \gg 1$ (dynamic perturbations are propagated more intensively than thermal ones), we can assume that heat transfer is concentrated in a narrow region at the wall, $y = R - r$, and that the thermal boundary layer of the liquid is plane.

Ignoring y in comparison with the radius R of the tube in (1), and introducing for convenience the dimensionless variables

$$v \equiv \frac{t - t_0}{t_w - t_0}, \quad X \equiv \frac{x}{D}, \quad Y \equiv \frac{y}{D}, \quad P \equiv \frac{D\langle w\rangle}{a}, \quad \omega(Y) \equiv \frac{w}{\langle w\rangle}$$

we can write Eq. (1) in the thermal boundary layer as

$$P\omega(Y)\frac{\partial v}{\partial X} = \frac{\partial^2 v}{\partial Y^2} \tag{2}$$

Here t_0 is the liquid temperature at the boundary of the thermal boundary layer, equal to the liquid temperature at the inlet to the channel, and $\langle w\rangle$ is the average flow velocity.

Equation (2) is approximate for a tube, but exact for a plane channel.

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Introducing the variable

$$\eta = F(Y)\, X^{-1/3} \tag{3}$$

we can reduce (2) to an ordinary differential equation

$$v'(\eta)\,[{}^4/_3\, P\omega(Y) F(Y) X^{-1/3} + F''(Y) X^{-1/3}] + v''(\eta)\,[F'(Y)]^2 X^{-2/3} = 0 \tag{4}$$

Here $F(Y) \sim P\omega(Y)$ is an, as yet, unknown function of y.

In general F(Y) is found from the equation of motion, using a rheological relation for the liquid under consideration.

Viscoelastic liquids evidently are the most general class of non-Newtonian media. Their rheological behavior must reflect in some form or another the particular features of the motion which are inherent in media which are structurally viscous, viscoplastic, etc. The effect of boundary-layer slip in viscoelastic media is very noticeable, as indicated by direct measurements of the slip of polymer melts at a wall using graphite capillaries [9]. It is also known [10] that the slip velocity of viscoelastic media can comprise approximately 80% of the shear velocity.

Consider the form of the function $\omega(Y)$ in the case when the viscoelastic properties of the medium are defined by the equations [11]

$$\pm w^{\cdot}\,[\tau + f(\tau)]^{-1} \equiv \varphi = \varphi_0 + \theta\,[\tau + f(\tau)] \tag{5}$$

$$f(\tau) \equiv p_{xx} - p_{rr} = -\gamma_e \tau_* \ln(1 - \tau/\tau_*) \tag{6}$$

Here $w^{\cdot} \equiv dw/dn$ is the velocity gradient in the direction of n; τ is the shear stress; θ is the instability coefficient of the elastic structure; φ_0 is the fluidity as $\tau \to 0$; γ_e is a dimensionless coefficient which is a measure of the appearance of the first difference of the normal stresses; τ_* is the critical value of the tangential shear stress (the saturation stress).

When $\gamma_e = 0$, we have motion in structurally viscous conditions, while when $\theta = 0$ we have motion with constant viscosity.

The distribution of the tangential stresses over the cross section of a viscoelastic flow in a tube under the assumption that $p_{rr} - p_{\varphi\varphi} \approx 0$ is determined from the equilibrium condition

$$\frac{\partial}{\partial x}(-p + p_{xx}) = \frac{1}{r}\frac{\partial}{\partial r}(r\tau), \quad \frac{\partial}{\partial r}(-p + p_{rr}) = 0 \tag{7}$$

Equations (7) are easily reduced to one equation

$$\frac{\partial}{\partial x}[-p(0, x) + p_{xx} - p_{rr}] = \frac{1}{r}\frac{\partial}{\partial r}(r\tau)$$

Here p(0, x) is the pressure on the channel axis.

From this equation we see that the profile of the tangential stresses over the cross section of a viscoelastic flow is linear.

For a tube

$$\tau = \tau_w \xi \quad (\xi = r/R) \tag{8}$$

where τ_w is the tangential stress at the wall.

We can write the expression for the average flow velocity in the tube as

$$1 = -\int_0^1 \xi^2 \frac{d\omega}{d\xi}\, d\xi \quad \left(\omega = \frac{w}{\langle w \rangle}\right) \tag{9}$$

Then, noting (5) and (9), we can write the expression for the dimensionless velocity as

$$\omega(\xi) = \left(\int_{\xi}^{1} \psi T^* \, d\xi + 2\zeta_S D^{-1}\right)\left(\int_{0}^{1} \xi^2 \psi T^* \, d\xi + 2\zeta_S D^{-1}\right)^{-1} \quad (10)$$

Here

$$T^* \equiv \xi\,[1 - \gamma_e (\Omega\xi)^{-1} \ln (1 - \Omega\xi)]\,[1 - \gamma_e \Omega^{-1} \ln (1 - \Omega)]^{-1}$$

$$\psi \equiv \frac{\varphi}{\varphi_w}, \quad \Omega \equiv \frac{\tau_w}{\tau_*} \lesssim 1$$

and φ_W is the fluidity of the liquid near the wall.

In (10) we have used the boundary condition permitting slip at the wall [12] in the form

$$w_S = \zeta_S \left(\frac{dw}{d\xi}\right)_{\xi=1}$$

where w_S is the slip velocity, ζ_S is the slip coefficient.

The term $2\zeta_S D^{-1}$, which takes into account the effect of slip, can be considered as a correction to the equation for continuous motion.

Substituting in (10) the corresponding expressions from (5), (6), and (8), and considering only small values of the parameter Ω, we obtain

for a circular tube

$$\omega(\xi) = 2\,[1 - \xi^2 + {}^2/_3\,\varepsilon\,(1 - \xi^3) + 4\zeta_S D^{-1}]\,[1 + 0.8\varepsilon + 8\zeta_S D^{-1}]^{-1},$$
$$\varepsilon = \frac{\theta\tau_*}{\varphi_0}\,\Omega\,(1 + \gamma_e) \quad (11)$$

for a plane channel

$$\omega(\xi) = 1.5\,[1 - \xi^2 + {}^2/_3\,\varepsilon\,(1 - \xi^3) + 4\zeta_S D^{-1}]\,[1 + 0.75\varepsilon + 6\zeta_S D^{-1}]^{-1} \quad (12)$$

Since $\xi = 1 - y/R$, $y \ll R$ in the thermal boundary layer, we can ignore the quadratic terms in (11) and (12).

Then for a circular tube we have

$$\omega(Y) = (8Y\,(1 + \varepsilon) + 8\zeta_S D^{-1}]\,[1 + 0.8\varepsilon + 8\zeta_S D^{-1}]^{-1} \quad (13)$$

Consequently,

$$\frac{w_S}{\langle w \rangle} = \left(1 + \frac{0.8\varepsilon D}{8\zeta_S} + \frac{D}{8\zeta_S}\right)^{-1}$$

Thus, in this case the slip velocity comprises a small proportion of the average velocity, if

$$0.125D\,(1 + 0.8\varepsilon) \gg \zeta_S$$

In accordance with (13) it is convenient to define F(Y) as

$$F(Y) = (Pm)^{1/3}\,(Y + S), \quad m = (1 + 0.8\varepsilon n^{-1} + 8S)^{-1}, \; S = \zeta_S/Dn,$$
$$n = 1 + \varepsilon \quad (14)$$

Let us obtain the solution for the case t_W = const. Substituting (13) and (14) in (4), we find

$$v''(\eta) + {}^8/_3\,\eta^2 v'(\eta) = 0 \quad (15)$$

The boundary conditions for (15) have the form

$$\eta = S\,(Pm/X)^{1/3} \equiv Z, \quad v = v_S \text{ as } Y \to 0$$
$$\eta \to \infty, \quad v = 1 \text{ as } Y \to \infty$$

The first boundary condition permits a discontinuity between the wall temperature and the temperature of the viscoelastic medium at the surface of the wall. The expression for the temperature discontinuity can be written in a form similar to the slip condition for the velocity

$$v_S = \frac{\varepsilon_T}{\partial} \frac{\partial v}{DY}\bigg|_{Y\to 0} \tag{16}$$

Here $\varepsilon_T \equiv \lambda / \alpha_S$ is the magnitude of the temperature discontinuity.

The solution of (15) is

$$v = \left[\int_z^\eta \exp\left(-\frac{8}{9}\eta^3\right)d\eta + v_S \int_\eta^\infty \exp\left(-\frac{8}{9}\eta^3\right)d\eta\right]\left[\int_z^\infty \exp\left(-\frac{8}{9}\eta^3\right)d\eta\right]^{-1} \tag{17}$$

Substituting (17) in (16), we have

$$v_S = L\left[L + \int_z^\infty \exp\left(-\frac{8}{9}\eta^3\right)d\eta\right]^{-1}, \quad L \equiv \varepsilon_T D^{-1}\left(\frac{Pm}{X}\right)^{1/3} \exp\left(-\frac{8}{9}Z^3\right) \tag{18}$$

We can transform (18) to the form

$$v_S = \left[1 + \left[\nu c\gamma \exp\left(-\frac{8}{9}Z^3\right)\right]^{-1} \alpha_S 3 \left(\frac{D^2 x}{Pm}\right)^{1/3} \int_z^\infty \exp\left(-\frac{8}{9}\eta^3\right)d\eta\right]^{-1}$$

from which we see that at very large Prandtl numbers σ the temperature discontinuity is an insignificant part of the temperature drop $t_W - t_0$ and it can be ignored. In (18) ν, c, and γ are respectively the kinematic viscosity, the specific heat, and the specific gravity of the liquid.

The local value of the Nusselt number is

$$N_x = \frac{\partial v}{\partial Y}\bigg|_{Y=0} = \left\{\frac{\varepsilon_T}{D} + \frac{1 - \Gamma(1/3, Z^3)/\Gamma(1/3)}{1.077\,(Pm/X)^{1/3}\exp(-8/9\,Z^3)}\right\}^{-1} \tag{19}$$

Figure 1 shows the results of calculations from this equation for the case $PmX^{-1} = 10$ (the curves 1-4 correspond to the values S = 0, 0.2, 0.6, 0.8). We see that slip may have a marked effect on heat transfer only in the region of small values of $\varepsilon_T D^{-1}$, i.e., when the Prandtl number is very large.

For the case t_W = const the thickness of the thermal boundary layer has the form

$$\delta_T = \varepsilon_T + 0.925\left[1 - \frac{\Gamma(1/3, Z^3)}{\Gamma(1/3)}\right]\left[\left(\frac{Pm}{xD^2}\right)^{1/3} \exp\left(-\frac{8}{9}Z^3\right)\right]^{-1} \tag{20}$$

We see that the boundary layer at the wall significantly affects the thickness of the thermal boundary layer.

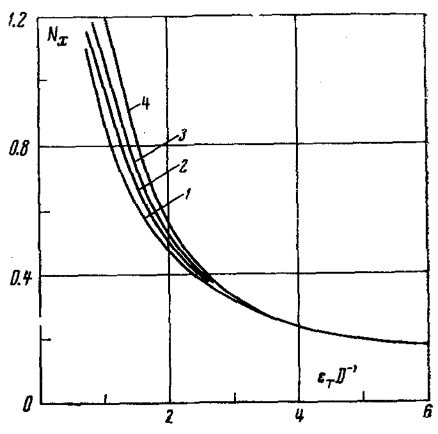

Fig. 1

Let us obtain the solution for the case q_W = const. Differentiating (2) with respect to Y and introducing the ratio of the densities of the thermal flux by means of the equation

$$q = \frac{\partial t}{\partial Y}\left(\frac{\partial t}{\partial Y}\right)^{-1}_{Y=0}$$

we have

$$8Pm\frac{\partial q}{\partial X} = \frac{\partial}{\partial Y}\left(\frac{1}{Y+S}\frac{\partial q}{\partial Y}\right) \tag{21}$$

By introducing the variable (3) we transform (21) to

$$q''(\eta) + \frac{8\eta^3 - 3}{3\eta}q'(\eta) = 0$$

the solution of which for the boundary conditions

$$Y \to 0, \quad \eta = Z, \quad q = 1; \quad Y \to \infty, \quad \eta \to \infty, \quad q = 0$$

has the form

$$q = \int_{\eta}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta \left[\int_{z}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta\right]^{-1}$$

Consequently,

$$\theta^* = \left(\frac{X}{mP}\right)^{1/3} \left[\int_{z}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta\right]^{-1} \left[\eta \int_{\eta}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta - \frac{3}{8} \exp\left(-\frac{8}{9}\eta^3\right)\right] + C$$

Here $\theta^* \equiv \lambda (t - t_0)/q_W D$ is the dimensionless temperature. The constant of integration is found from the condition (16). After some calculations we find that

$$\theta^* = \left(\frac{X}{mP}\right)^{1/3} \left[\int_{z}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta\right]^{-1} \left[\eta \int_{\eta}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta - \frac{3}{8} \exp\left(-\frac{8}{9}\eta^3\right) + \left(\frac{\varepsilon_T}{D} - S\right) \int_{z}^{\infty} \eta \exp\left(-\frac{8}{9}\eta^3\right) d\eta + \frac{3}{8} \exp\left(-\frac{8}{9} Z^3\right)\right] \quad (22)$$

Defining the local Nusselt number

$$N_x = \frac{q_w D}{(t_w - t_0)\lambda}$$

we obtain the following from (22):

$$N_x = \left\{\frac{\varepsilon_T}{D} - S + \frac{\exp(-{}^8/_9 Z^3)}{1.29\,(mP/X)^{1/3}\,[1 - \Gamma({}^2/_3, Z^3)/\Gamma({}^2/_3)]}\right\}^{-1} \quad (23)$$

Similar expressions are obtained for a plane channel with the only difference that instead of the coefficients 1.077 (t_W = const) and 1.29 (q_W = const), we have to use 0.978 and 1.175, respectively. Then the parameter m is determined from the equation

$$m = n\,(1 + 0.75\varepsilon + 6\zeta_S D^{-1})^{-1}$$

In conclusion it should be noted that in the absence of velocity and temperature discontinuities at the heat-transfer surface, and also when $\gamma_e = 0$, the heat-transfer expressions become the familiar relations for structurally viscous [13], and as $m \to 1$ ($\theta \approx 0$) for ordinary Newtonian liquids [14].

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